

# Maximal selectivity for orders in fields

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## Abstract

If  $\mathfrak{H} \subseteq \mathfrak{D}$  are two orders in a central simple algebra  $\mathfrak{A}$  with  $\mathfrak{D}$  of maximal rank, the theory of representation fields describes the set of spinor genera of orders in the genus of  $\mathfrak{D}$  representing the order  $\mathfrak{H}$ . When  $\mathfrak{H}$  is contained in a maximal subfield of  $\mathfrak{A}$  and the dimension of  $\mathfrak{A}$  is the square of a prime  $p$ , the proportion of spinor genera representing  $\mathfrak{H}$  has the form  $r/p$ , in fact, when the representation field exists, this proportion is either 1 or  $1/p$ . In the later case the order  $\mathfrak{H}$  is said to be selective for the genus. The condition for selectivity is known when  $\mathfrak{D}$  is maximal and also when  $p = 2$  and  $\mathfrak{D}$  is an Eichler order. In this work we describe the orders  $\mathfrak{H}$  that are selective for at least one genus of orders of maximal rank in  $\mathfrak{A}$ .

## 1 Introduction

Let  $K$  be a number field. Let  $\mathfrak{A}$  be a central simple algebra (CSA) over  $K$ , and let  $\mathfrak{D}$  be an order of maximal rank in  $\mathfrak{A}$ . Let  $\Sigma$  be the spinor class field for the genus  $\mathbb{O}$  of  $\mathfrak{D}$  as defined in [1]. In particular,  $\Sigma/K$  is an abelian extension such that the spinor genera in  $\mathbb{O}$  can be described in terms of the Galois group  $\mathcal{G} = \text{Gal}(\Sigma/K)$ . More precisely, there exists a map

$$\rho : \mathbb{O} \times \mathbb{O} \rightarrow \mathcal{G},$$

Such that  $\mathfrak{D}'$  belongs to the spinor genus  $\text{spin}(\mathfrak{D})$  if and only if  $\rho(\mathfrak{D}, \mathfrak{D}') = \text{Id}_{\Sigma}$ . Furthermore, the map  $\rho$  satisfies  $\rho(\mathfrak{D}, \mathfrak{D}'') = \rho(\mathfrak{D}, \mathfrak{D}')\rho(\mathfrak{D}', \mathfrak{D}'')$  for any triple  $(\mathfrak{D}, \mathfrak{D}', \mathfrak{D}'') \in \mathbb{O}^3$  ([1], §3). Note that when strong approximation

applies to the algebraic group  $\mathrm{SL}_1(\mathfrak{A})$ , spinor genera coincide with conjugacy classes [1]. All of the above generalize to arbitrary global fields. This is done in ([3], §2) when  $\mathfrak{D}$  is maximal, although the last hypothesis is not actually used there. In fact, everything that follows holds in the function field case as much as in the number field case. It also holds for  $S$ -orders for an arbitrary set of places  $S$  containing the archimedean places if any.

For any suborder  $\mathfrak{H}$  of  $\mathfrak{D}$ , we can define two intermediate subfields:

1. The upper representation field  $F = F^-(\mathfrak{D}|\mathfrak{H})$  for the pair  $(\mathfrak{D}|\mathfrak{H})$  is the smallest subfield of  $\Sigma$  containing  $K$  such that the order  $\mathfrak{H}$  is represented by the spinor genus  $\mathrm{Spin}(\mathfrak{D}')$  for every order  $\mathfrak{D}' \in \mathbb{O}$  satisfying  $\rho(\mathfrak{D}, \mathfrak{D}')|_F = \mathrm{Id}_F$ .
2. The lower representation field  $F = F_-(\mathfrak{D}|\mathfrak{H})$  is the largest subfield of  $\Sigma$  such that  $\mathfrak{H} \subseteq \mathfrak{D}'$  implies  $\rho(\mathfrak{D}, \mathfrak{D}')|_F = \mathrm{Id}_F$ .

When  $F_-(\mathfrak{D}|\mathfrak{H}) = F^-(\mathfrak{D}|\mathfrak{H})$ , we call this field the representation field  $F(\mathfrak{D}|\mathfrak{H})$  for the pair  $(\mathfrak{D}|\mathfrak{H})$ . In this case we say that the representation field exists. The existence of a representation field  $F$  for  $\mathfrak{H}$  implies that the proportion of conjugacy classes in  $\mathbb{O}$  representing  $\mathfrak{H}$  is  $[F : K]^{-1}$ . This fact was first studied by Chevalley [6] when  $\mathfrak{A}$  is a matrix algebra of arbitrary dimension,  $\mathfrak{D}$  is a maximal order, and  $\mathfrak{H}$  is the maximal order in a maximal subfield of  $\mathfrak{A}$ . Later computations for  $F(\mathfrak{D}|\mathfrak{H})$  can be summarized in the following table:

Year	Ref.	$\mathfrak{A}$ (CSA)	$\mathfrak{D}$	$\Sigma/K$	$\mathfrak{H}$	$K\mathfrak{H}$
1936	[6]	matrix	max.		max.	field (max.)
1999	[7]	quaternion	max.			commutative
2003	[1]	NPR	max.		max.	field (maximal)
2004	[8],[5]	quaternion	EO			commutative
2004	[5] <sup>e</sup>	quaternion				commutative
2008	[2] <sup>e</sup>		max.	$\mathcal{G}^2 = \{\mathrm{id}\}$		
2008	[2] <sup>c</sup>		max.	$\mathcal{G}^2 \neq \{\mathrm{id}\}$	GEO	$= \mathfrak{A}$
2010	[11]	prime degree	max.			field
2011	[10]	quaternion		unramified		commutative
2011	[3]		max.			commutative
2011	[4]		max.		LCMO	CSA

NPR above stands for *no partial ramification*, a weaker condition than prime degree, while LCMO means *locally cyclic or maximal order*, and (G)EO means *(generalized) Eichler order*. Here  $[n]^c$  denotes a counter-example, i.e.,

an example where the representation field fails to exist, while  $[n]^e$  denotes an existential result. The proof of the main result in [3] seems to be easy to generalize to other families of suborders. For example, it is very simple now to write a general formula for the representation field when  $K\mathfrak{H}$  is contained in a quaternion algebra. However, the condition that  $\mathfrak{D}$  is maximal is essential in this computations and a generalization to arbitrary orders  $\mathfrak{D}$  of maximal rank seems unlikely at this point.

In this paper we focus on the case where  $\mathfrak{H}$  is an order of maximal rank in a maximal subfield of  $\mathfrak{A}$ . Instead of trying to give a general formula for all representation fields  $F(\mathfrak{D}|\mathfrak{H})$ , we focus on the maximal possible representation field  $F_M(\mathfrak{A}|\mathfrak{H}) = \max_{\mathfrak{D} \subseteq \mathfrak{A}} F(\mathfrak{D}|\mathfrak{H})$ , where  $\mathfrak{D}$  runs over the set of all orders of maximal rank for which  $F(\mathfrak{D}|\mathfrak{H})$  is defined. It follows easily from formula (7) in §4.2 of [1] that  $F^-(\mathfrak{D}|\mathfrak{H}) \subseteq L$  for every order  $\mathfrak{D}$  of maximal rank, and therefore also  $F_M(\mathfrak{A}|\mathfrak{H}) \subseteq L$ , as long as this maximum exists. Here we give a formula for  $F_M(\mathfrak{A}|\mathfrak{H})$  whenever  $\sqrt{\dim_K(\mathfrak{A})} = p$  is a prime. In particular we prove the existence of  $F_M(\mathfrak{A}|\mathfrak{H})$  in this case. For extensions  $L/K$  of prime degree, an order of maximal rank in  $L$  is selective (in the sense defined in [7] on [11]) for some genus of orders of maximal rank in  $\mathfrak{A}$  if and only if  $F_M(\mathfrak{A}|\mathfrak{H}) = L$  (Prop. 3.1). Since the even-dimensional and odd-dimensional cases are different, we state our results in the next two theorems:

**Theorem 1.1.** *Let  $\mathfrak{A}$  be a quaternion algebra and  $L \cong K(\sqrt{d})$  a maximal subfield. Then, for any order  $\mathfrak{H}$  of maximal rank in  $L$ , we have  $F_M(\mathfrak{A}|\mathfrak{H}) = L$  if and only if  $\mathfrak{A}$  ramifies at exactly the same set of finite primes as the algebra  $(\frac{-1,d}{K})$ . Otherwise we have  $F_M(\mathfrak{A}|\mathfrak{H}) = K$ .*

When  $L/K$  is a Galois extension, we say that an order  $\mathfrak{H} \subseteq L$  is asymmetrical at a non-split place  $\wp$  if  $\sigma(\mathfrak{H}_\wp) \neq \mathfrak{H}_\wp$  for some element  $\sigma \in \text{Gal}(L/K)$ .

**Theorem 1.2.** *Assume that  $\dim_K(\mathfrak{A}) = p^2$  where  $p$  is an odd prime, and let  $L \subseteq \mathfrak{A}$  be a maximal subfield. Then, for any order  $\mathfrak{H}$  of maximal rank in  $L$ , we have  $F_M(\mathfrak{A}|\mathfrak{H}) = L$  if and only if  $L/K$  is Galois and the order  $\mathfrak{H}$  is asymmetrical at every finite place that is ramified for  $\mathfrak{A}$ . If this condition is not satisfied we have  $F_M(\mathfrak{A}|\mathfrak{H}) = K$ .*

## 2 A continuity principle

In all of this section,  $K$  is a local field and  $\mathfrak{A}$  is a central simple  $K$ -algebra. We denote by  $x \mapsto |x|$  the absolute value on  $\mathfrak{A}$  or  $K$ . Note that  $\mathfrak{A}$  is locally

compact since it is a finite dimensional vector space over the locally compact field  $K$ .

**Lemma 2.1.** *Let  $\mathfrak{A}$  and  $K$  be as above. Then the conjugation stabilizer of a maximal order is compact in  $\mathfrak{A}^*/K^*$ .*

*Proof.* Assume first that  $\mathfrak{A}$  be a division algebra. We claim that  $\mathfrak{A}^*/K^*$  is compact. The result follows from the claim since a division algebra has a unique maximal order ([12], Ch. 1, Thm. 6). Since  $e = \left[ |\mathfrak{A}^*| : |K^*| \right]$  is finite, it suffices to observe that the kernel of the absolute value is  $N = \overline{B(0; 1)} - B(0; 1)$ , where  $B(0; 1)$  is the open ball in  $\mathfrak{A}$  centered at 0, which is a compact set.

Assume now that  $\mathfrak{A} \cong \mathbb{M}_m(\mathfrak{A}_0)$  for some division algebra  $\mathfrak{A}_0$ . The conjugation-stabilizer of a maximal order  $\mathfrak{D}$  is  $\mathfrak{D}^*\mathfrak{A}_0^*/K^*$ , whence the conclusion follows since  $\mathfrak{A}_0^*/K^*$  is compact by the previous result and  $\mathfrak{D}^*$  is compact since it is closed in the compact set  $\mathfrak{D}$ .  $\square$

In any metric space  $(X, d)$  we define for every pair of subsets  $A$  and  $B$  of  $X$

$$\rho(A, B) = \sup_{a \in A} d(a, B).$$

When  $B$  is closed, we have  $\rho(A, B) = 0$  if and only if  $A \subseteq B$ . Note that  $\rho$  is not a metric, since it is not symmetric, but  $\hat{\rho}(A, B) = \rho(A, B) + \rho(B, A)$  is a metric on the collection of compact subsets of  $X$  called the Hausdorff metric.

In all that follows, for every pair of orders  $\mathfrak{D}$  and  $\mathfrak{H}$  in  $\mathfrak{A}$  we denote

$$H(\mathfrak{D}|\mathfrak{H}) = \{N(u) | u \in \mathfrak{A}^*, u^{-1}\mathfrak{H}u \subseteq \mathfrak{D}\}, \quad H(\mathfrak{H}) = H(\mathfrak{H}|\mathfrak{H}),$$

where  $N : \mathfrak{A}^* \rightarrow K^*$  is the reduced norm.

**Proposition 2.1.** *Let  $\mathfrak{A}$  be a central simple algebra over the local field  $K$ . Assume the order  $\mathfrak{H}$  is contained in finitely many maximal orders, and let  $\{\mathfrak{D}_t\}_{t \in \mathbb{N}}$  be a sequence of orders such that  $\rho(\mathfrak{D}_t, \mathfrak{D}) \xrightarrow{t \rightarrow \infty} 0$ . Then, in the set theoretical sense:*

$$\limsup_{t \rightarrow \infty} H(\mathfrak{D}_t|\mathfrak{H}) \subseteq H(\mathfrak{D}|\mathfrak{H}).$$

*Proof.* It suffices to prove that if  $a \in H(\mathfrak{D}_t|\mathfrak{H})$  for infinitely many values of  $t$ , then  $a \in H(\mathfrak{D}|\mathfrak{H})$ . The hypotheses imply  $a \in N(y_t)K^{*2}$  for some  $y_t$  satisfying  $y_t\mathfrak{H}y_t^{-1} \subseteq \mathfrak{D}_t$ . Let  $\mathfrak{D}'$  be a maximal order containing  $\mathfrak{D}$ . Since  $\mathfrak{D}'$  is open,

then  $\mathfrak{D}_t \subseteq \mathfrak{D}'$  for  $t$  sufficiently large. If  $y_t \mathfrak{H} y_t^{-1} \subseteq \mathfrak{D}_t$ , then  $\mathfrak{H} \subseteq y_t^{-1} \mathfrak{D}_t y_t$ , whence  $\mathfrak{H} \subseteq y_t^{-1} \mathfrak{D}' y_t$  for  $t$  sufficiently large. It follows that the set of maximal orders  $\{y_t^{-1} \mathfrak{D}' y_t\}_t$  is finite. Write  $\bar{x}$  for the class in  $\mathfrak{A}^*/K^*$  of an element  $x \in \mathfrak{A}^*$ . As the stabilizer of  $\mathfrak{D}'$  in  $\mathfrak{A}^*/K^*$  is compact, the sequence  $\{\bar{y}_t\}_{t \in \mathbb{N}}$  is contained in a compact set, whence, by taking a subsequence if needed, we can assume it is convergent in  $\mathfrak{A}^*/K^*$  to an element  $\bar{y}$ . In particular,  $y \mathfrak{H} y^{-1} \subseteq \mathfrak{D}$ , and  $N(y_t) \in N(y) K^{*2}$  for  $t$  sufficiently large. We conclude that  $a \in N(y) K^{*2}$ , and the result follows.  $\square$

**Corollary 2.1.** *Let  $\mathfrak{H}$  be an order contained in finitely many maximal orders. Then there exists  $\epsilon = \epsilon(\mathfrak{H})$  such that whenever  $\mathfrak{H} \subseteq \mathfrak{D}$  with  $\rho(\mathfrak{D}, \mathfrak{H}) \leq \epsilon$ , then  $H(\mathfrak{D}|\mathfrak{H}) = H(\mathfrak{H})$ .*

*Proof.* Note that the set of quadratic classes is finite, so one inclusion follows from the previous lemma. The opposite inclusion is immediate, since  $H(\mathfrak{D}|\mathfrak{H}) = H(\mathfrak{D}|\mathfrak{H})H(\mathfrak{H})$  by the general theory [1].  $\square$

**Proposition 2.2.** *Assume  $\mathfrak{A}$  is a matrix algebra. If  $L = K\mathfrak{H}$  is a maximal subfield of  $\mathfrak{A}$ , then  $\mathfrak{H}$  is contained in finitely many maximal orders of  $\mathfrak{A}$ .*

*Proof.* Since every pair of embeddings of  $L$  into  $\mathfrak{A}$  are conjugate, we can identify  $L$  with its natural image in  $\text{Aut}_K(L) \cong \text{Aut}_K(K^n) \cong \mathfrak{A}$ . The  $\mathfrak{H}$ -invariant lattices in  $K^n$  correspond to fractional  $\mathfrak{H}$ -ideals in  $L$ . It suffices, therefore, to prove that  $K^*$  acts on the set of fractional  $\mathfrak{H}$ -ideals with finitely many orbits. Let  $\Lambda$  be a fractional ideal. Multiplying by an element of  $K^*$  if needed we can assume that  $\Lambda \subseteq \mathcal{O}_L$ , but  $\Lambda$  is not contained in  $\pi_K \mathcal{O}_L$  for a uniformizing parameter  $\pi_K$  of  $K$ , i.e., there exist some element  $u \in \Lambda \setminus \pi_K \mathcal{O}_L$ . Since  $\mathcal{O}_L$  is a valuation ring, we have  $\pi_K \mathcal{O}_L \subseteq u \mathcal{O}_L$ . Since  $\mathfrak{H}$  has maximal rank in  $L$ ,  $\pi_K^N \mathcal{O}_L \subseteq \mathfrak{H}$  for some  $N$ , whence  $\pi_K^{N+1} \mathcal{O}_L \subseteq \pi_K^N \mathcal{O}_L u \subseteq \mathfrak{H} \Lambda = \Lambda$ . It follows that  $\pi_K^{N+1} \mathcal{O}_L \subseteq \Lambda \subseteq \mathcal{O}_L$  and the result follows.  $\square$

**Corollary 2.2.** *Let  $\mathfrak{H}$  be an order of maximal rank in a maximal subfield of  $\mathfrak{A}$ . Then there exists  $\epsilon = \epsilon(\mathfrak{H})$  such that whenever  $\mathfrak{H} \subseteq \mathfrak{D}$  with  $\rho(\mathfrak{D}, \mathfrak{H}) \leq \epsilon$ , then  $H(\mathfrak{D}|\mathfrak{H}) = H(\mathfrak{H})$ .*  $\square$

**Example.** Assume  $\mathfrak{A}$  is a split quaternion algebra and  $\mathfrak{H}$  is an order in a maximal unramified subfield  $L$ . It is proved in [8] that  $H(\mathfrak{D}|\mathfrak{H}) = K^*$  whenever  $\mathfrak{D}$  is an Eichler order representing an order in  $L$  strictly containing  $\mathfrak{H}$ . This result does not generalize to arbitrary orders of maximal rank. For example if  $\mathfrak{D}_k \xrightarrow{k \rightarrow \infty} \mathcal{O}_L$  in the Hausdorff metric, then for  $k$  big enough we have  $H(\mathfrak{D}_k|\mathfrak{H}) \subseteq H(\mathcal{O}_L|\mathfrak{H}) = H(\mathcal{O}_L) = \mathcal{O}_K^* K^{*2}$  according to the computations in §3.

### 3 Computation of $F_M$

In all that follows,  $K$  is a global field,  $L/K$  is a field extension of prime degree  $p$  and  $\mathfrak{A}$  is a  $p^2$ -dimensional central simple algebra containing a copy of  $L$ . We let  $J_K$  be the idele group of  $K^*$ ,  $\mathfrak{A}_{\mathbb{A}}$  the adelization of the algebra  $\mathfrak{A}$ , and  $N : \mathfrak{A}_{\mathbb{A}}^* \rightarrow J_K$  the adelic reduced norm. For every pair of global orders  $\mathfrak{H} \subseteq \mathfrak{D}$  we define  $H(\mathfrak{D}|\mathfrak{H})$  as the set

$$\{N(u) | u \in \mathfrak{A}_{\mathbb{A}}^*, u^{-1}\mathfrak{H}u \subseteq \mathfrak{D}\} = \left[ \prod_{\varphi \notin S} H(\mathfrak{D}_{\varphi}|\mathfrak{H}_{\varphi}) \times \prod_{\varphi \in S} N(\mathfrak{A}_{\varphi}^*) \right] \cap J_K,$$

where  $H(\mathfrak{D}_{\varphi}|\mathfrak{H}_{\varphi})$  is defined as in the preceding section, and  $H(\mathfrak{H}) = H(\mathfrak{H}|\mathfrak{H})$ . Define the abelian extension  $F_0(\mathfrak{A}|\mathfrak{H})$  of  $K$  as the class field corresponding to  $K^*H(\mathfrak{H})$ . Note that  $H(\mathfrak{H})$  is a group since it is the image under the reduced norm of the conjugation-stabilizer of  $\mathfrak{H}$ . It is immediate from the general theory that  $H(\mathfrak{H})H(\mathfrak{D}|\mathfrak{H}) = H(\mathfrak{D}|\mathfrak{H})$  for any order  $\mathfrak{D}$  of maximal rank, whence in particular  $F^-(\mathfrak{D}|\mathfrak{H}) \subseteq F_0(\mathfrak{A}|\mathfrak{H})$ . It follows that an order  $\mathfrak{H}$  such that  $F_0(\mathfrak{A}|\mathfrak{H}) = K$  cannot be selective for any genus.

Let  $\varphi$  be a non-split place for  $L/K$ . Note that the local conjugation-stabilizer  $N_{\varphi}$  of  $\mathfrak{H}_{\varphi}$  fits into a short exact sequence  $L_{\varphi}^* \hookrightarrow N_{\varphi} \twoheadrightarrow \Gamma_{\varphi}$ , where  $\Gamma_{\varphi}$  is contained in the Galois group  $\text{Gal}(L/K)$ . Furthermore, by Skolem-Noether's Theorem,  $\Gamma_{\varphi}$  is trivial only in the following cases:

1.  $L_{\varphi}/K_{\varphi}$  is not Galois.
2.  $\mathfrak{H}_{\varphi}$  is asymmetrical.

In any other case  $\Gamma_{\varphi}$  is a cyclic group of order  $p$ . Note that in particular  $H(\mathfrak{H}) \supseteq N_{L/K}(J_L)$ , whence it follows that  $F_0(\mathfrak{A}|\mathfrak{H}) \subseteq L$ .

**Proposition 3.1.** *Let  $\mathfrak{A}$  be a central simple  $p^2$ -dimensional  $K$ -algebra, where  $p$  is a prime. If  $L = K\mathfrak{H}$  is a maximal subfield of  $\mathfrak{A}$ , then there exists an order of maximal rank  $\mathfrak{D}$  in  $\mathfrak{A}$  satisfying  $F(\mathfrak{D}|\mathfrak{H}) = F_0(\mathfrak{A}|\mathfrak{H})$ . In particular,  $F_M(\mathfrak{A}|\mathfrak{H}) = F_0(\mathfrak{A}|\mathfrak{H})$  and  $\mathfrak{H}$  is selective for some genus of maximal orders of maximal rank in  $\mathfrak{A}$  if and only if  $F_0(\mathfrak{A}|\mathfrak{H}) \neq K$ .*

*Proof.* If  $L/K$  is not Galois, the contention  $F^-(\mathfrak{D}|\mathfrak{H}) \subseteq L$  shows that  $F(\mathfrak{D}|\mathfrak{H})$  is defined and equals  $K$  for any order  $\mathfrak{D}$  of maximal rank, and  $F_0(\mathfrak{A}|\mathfrak{H}) = K$  for the same reason, whence we can assume that  $L/K$  is Galois. Let  $T$  be the set of all finite places  $\varphi$  satisfying one of the following conditions:

1.  $\mathfrak{A}_\varphi$  is ramified.
2.  $L/K$  is inert at  $\varphi$  and  $\mathfrak{H}_\varphi$  is not maximal in  $L_\varphi$ .
3.  $L/K$  is ramified at  $\varphi$ .

For any  $\varphi \notin T$  we choose  $\mathfrak{D}_\varphi$  maximal. Let  $\mathbb{H}_\varphi$  be the residual algebra defined in [3]. When  $L/K$  splits at  $\varphi$ , every representation of the residual algebra  $\mathbb{H}_\varphi$  has dimension 1, so we have  $H_\varphi(\mathfrak{D}|\mathfrak{H}) = K_\varphi^*$  (Lemma 3.4 in [3]). When  $L/K$  is inert at  $\varphi$ ,  $\mathfrak{A}_\varphi$  is unramified, and  $\mathfrak{H}_\varphi$  is maximal in  $L_\varphi$ , every representation of the residual algebra  $\mathbb{H}_\varphi$  has dimension  $p$  and therefore  $H_\varphi(\mathfrak{D}|\mathfrak{H}) = \mathcal{O}_\varphi^* K^{*p}$  (Lemma 3.4 in [3]). In any case

$$H_\varphi(\mathfrak{H}) \subseteq H_\varphi(\mathfrak{D}|\mathfrak{H}) = N_{L_\varphi/K_\varphi}(L_\varphi^*) \subseteq H_\varphi(\mathfrak{H})$$

at all finite places  $\varphi \notin T$ . For the places  $\varphi \in T$ , we choose  $\mathfrak{D}_\varphi$  satisfying the conclusion of Corollary 2.2. The result follows. The last statement is a consequence of this and the discussion at the beginning of the section.  $\square$

**Proof of Theorem 1.1** Note that every order in a quadratic extension is symmetric. Assume  $L \subseteq \mathfrak{A}$  and let  $c \in \mathfrak{A}$  be a pure quaternion satisfying  $cac^{-1} = \bar{a}$  for every  $a \in L$ . Then  $r = c^2$  is a reduced norm from  $L_\varphi$  if and only if  $\mathfrak{A}_\varphi$  splits. Assume first that  $\mathfrak{A}_\varphi$  is a matrix algebra. Then the reduced norm  $N(c) = -r$  is in  $N_{L_\varphi/K_\varphi}(L_\varphi^*)$  if and only if  $-1 \in N_{L_\varphi/K_\varphi}(L_\varphi^*)$ . Assume next that  $\mathfrak{A}_\varphi$  is a division algebra. Then,  $F_\varphi$  is a field, whence  $N_{L_\varphi/K_\varphi}(L_\varphi^*)$  is a subgroup of index 2 in  $K_\varphi^*$ . It follows that  $-r \in N_{L_\varphi/K_\varphi}(L_\varphi^*)$  if and only if  $-1 \notin N_{L_\varphi/K_\varphi}(L_\varphi^*)$ . The result follows in either case.  $\square$

**Proof of Theorem 1.2** As in the proof of Proposition 3.1 we can assume that  $L/K$  is Galois. Let  $\sigma$  be a generator of the Galois group  $\text{Gal}(L/K)$  and assume  $L \subseteq \mathfrak{A}$ . Fix a local place  $\varphi$  and let  $c \in \mathfrak{A}_\varphi$  be an element satisfying  $cac^{-1} = \sigma(a)$  for every  $a \in L_\varphi$ . Such a  $c$  exists by Skolem-Noether's Theorem. For any generator  $v$  of  $L/K$ , we have  $c^r v = \sigma^r(v) c^r$ , whence the powers of  $c$  are eigenvectors of the map  $x \mapsto xv$ , and therefore linearly independent over  $L$ . It follows that  $L$  and  $c$  generate  $\mathfrak{A}$  and therefore  $r = c^p \in K$ , since it is central. Note that  $r$  is a reduced norm from  $L_\varphi$  if and only if  $\mathfrak{A}_\varphi$  splits ([9], Prop 30.6). The result follows now since  $N(c) = r$ .  $\square$

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